Math Circles Grade 11/12 Session 2

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Limits

The graph represents

$$\lim_{x \to 5} f(x) = L$$

limit of f(x) as $x \to 5$ is L

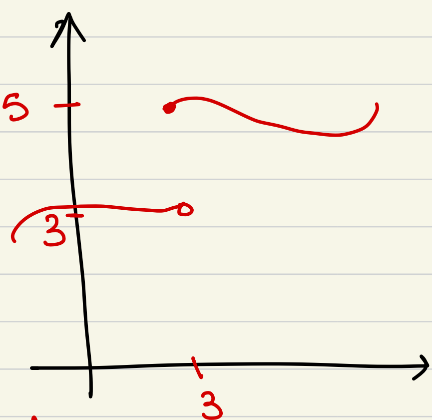
left hand limit $\rightarrow$ example $\lim_{x \to 5^-} x+4$ General $\lim_{x \to a^-} f(x)$

right hand limit $\rightarrow \lim_{x \to 5^+} 3x$ $\lim_{x \to a^+} f(x)$

What if the function has different limit values from left and right side?

$\lim_{x \to 3^-} f(x) = 3$ (Limit from left side)

$\lim_{x \to 3^+} f(x) = 5$ (Limit from right side)



Since $\lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$

$\therefore \lim_{x \to 3} f(x)$ does not exist.

Finding limits

While finding limits is a vast topic, we will cover basic limits only, to give you a rough idea of rate of change calculations.

$$\lim_{x \to 3} 5x = 5(3) = 15$$

$$\lim_{x \to 10} \frac{x+5}{x+10} = \frac{10+5}{10+10} = \frac{15}{20} = \frac{3}{4}$$

$$\lim_{x \to 5} \frac{x-5}{x^2-5^2} = \lim_{x \to 5} \frac{\cancel{x-5}}{\cancel{(x-5)}(x+5)} = \lim_{x \to 5} \frac{1}{x+5}$$

$$= \frac{1}{5+5} = \frac{1}{10}$$

$\left(\frac{0}{0}\right)$ form

Rate of change or derivative

Let $y = f(x)$ be a function.

The rate of change of $f(x)$ at $x = 4$ is given as

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{x+h-x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

limit as $h \to 0$

It is denoted by $\frac{df}{dx}$ and $f'(x)$.

That is $\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Finding rate of change of a function with 1 dimensional input

Let $f(x) = x^2$. Find $\frac{df}{dx}$.

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} + h^2 + 2xh - \cancel{x^2}}{h}$$

$$= \lim_{h \to 0} \frac{h^2 + 2xh}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(h + 2x)}{\cancel{h}}$$

$$= \lim_{h \to 0} h + 2x$$

$$= 0 + 2x$$

$$\boxed{= 2x}$$

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

$$\frac{d}{dx}(k) = 0 \text{ for any } k \in \mathbb{R}$$

Some properties of derivatives

(i) $\frac{d}{dx}(f \pm g) = \frac{df}{dx} \pm \frac{dg}{dx}$

(ii) $\frac{d}{dx}(f \cdot g) = \left(\frac{df}{dx}\right) g(x) + f(x)\left(\frac{dg}{dx}\right)$ Product rule

eg. $\frac{d}{dx}(x^2 + 2) = \frac{d}{dx}(x^2) + \frac{d}{dx}(2)$

$= 2x + 0$

$= 2x$

(iii) $\frac{d}{dx}(kf(x)) = k\frac{d}{dx}(f(x))$

eg. $\frac{d}{dx}(3x^2) = 3\frac{d}{dx}(x^2) = 3(2x) = 6x$

eg. $\frac{d}{dx}\left(x^3(x^2+3)\right) = \frac{d}{dx}(x^5 + 3x^3)$

$= \frac{d}{dx}(x^5) + \frac{d}{dx}(3x^3)$

$= 5x^4 + 3\frac{d}{dx}(x^3)$

$= 5x^4 + 3(3x^2)$

$= 5x^4 + 9x^2$

$$\frac{d}{dx}(f \cdot g) = \left(\frac{df}{dx}\right) g(x) + f(x)\left(\frac{dg}{dx}\right)$$ Product rule

eg: $$\frac{d}{dx}\left(x^3(x^2+3)\right) = \left(\frac{d(x^3)}{dx}\right)(x^2+3) + x^3\left(\frac{d(x^2+3)}{dx}\right)$$

$f(x) = x^3$, $g(x) = x^2+3$

$$= (3x^2)(x^2+3) + x^3(2x+0)$$

$$= 3x^4 + 9x^2 + 2x^4$$

$$= 5x^4 + 9x^2$$

(iv) $$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{g(x)\frac{df}{dx} - f(x)\frac{dg}{dx}}{(g(x))^2}$$ (Quotient rule)

eg: $$\frac{d}{dx}\left(\frac{x^3+3}{x^5}\right) = \frac{x^5 \cdot \frac{d(x^3+3)}{dx} - (x^3+3)\frac{d(x^5)}{dx}}{(x^5)^2}$$

$f(x) = x^3+3$
$g(x) = x^5$

$$= \frac{x^5(3x^2) - (x^3+3)(5x^4)}{x^{10}}$$

$$= \frac{3x^7 - 5x^7 - 15x^4}{x^{10}}$$

$$= \frac{-2x^7 - 15x^4}{x^{10}} = \frac{-2}{x^3} - \frac{15}{x^6}$$

$$= -2x^{-3} - 15x^{-6}$$

Finding rate of change of a function with more than 1 input variable

Let $f(x,y) = x^2 + y^2 + 2xy$.

We need the concept of partial derivatives for this case.

[This is calculated by assuming y is a constant]

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 + y^2 + 2(x+h)y - x^2 - y^2 - 2xy}{h}$$

$$= \lim_{h \to 0} \frac{x^2 + h^2 + 2xh + 2xy + 2hy - x^2 - 2xy}{h}$$

$$= \lim_{h \to 0} \frac{h^2 + 2xh + 2hy}{h}$$

$$= \lim_{h \to 0} \frac{h(h + 2x + 2y)}{h}$$

$$= \lim_{h \to 0} h + 2x + 2y$$

$$= 0 + 2x + 2y$$

$$= 2x + 2y.$$

This is calculated by assuming x is a constant

Similarly, $\frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x,y)}{h}$

$= 2x + 2y$

Then the rate of change of f is called gradient of f and is denoted by ∇f.

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}$$

In the above case, $\nabla f = \begin{bmatrix} 2x+2y \\ 2x+2y \end{bmatrix}$

The exponential function

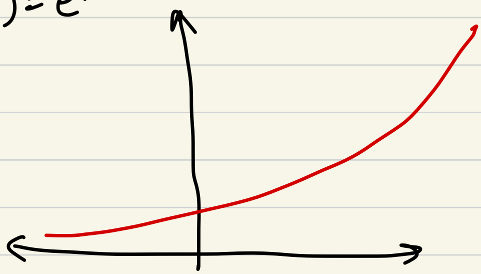
a^x where a is a constant and x is a variable

Special case:

$a = e \approx 2.71828$

e is a transcendental number

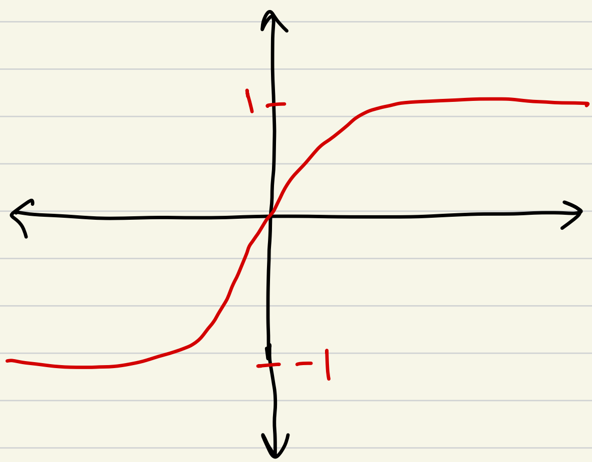
$f(x) = e^x$



$f(x) = e^{-x}$

The sigmoid function

$$\sigma(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{1+e^x}$$



bounded and strictly increasing function

ReLU function

↓ Rectified linear unit

$$f(x) = \max(0, x)$$

eg. $f(3) = 3$

$f(-4) = 0$

Session 2 Questions

Q1- Evaluate the following limits:

(a) $\lim_{x \to 5} x^2+2x+3$ (b) $\lim_{x \to 0} \frac{x^2+x}{x}$

(c) $\lim_{x \to 2} \frac{7x-14}{x^2-4x+4}$ (d) $\lim_{x \to -3} \frac{6x+18}{(x+4)(x+3)}$

Q2- Let

$$f(x) = \begin{cases} x & \text{for } x < 0 \\ x-2 & \text{for } x \geq 0 \end{cases}$$

Evaluate $\lim_{x \to 2} f(x)$ and $\lim_{x \to 0} f(x)$.

Q3- Find $\frac{df}{dx}$ if (a) $f(x) = x^2+2x+4$.

(b) $f(x) = x^3$

Q4- Find ∇f if $f(x,y) = x^3+2x^2+3x^2y+3xy+4$.

Q5- Find ∇f if $f(x,y,z) = x+y+z^2x$

Q6- Find the derivative of sigmoid function.

(Assume that $\lim_{h \to 0} \frac{e^h-1}{h} = 1$)

Q7- Find $\frac{d}{dx}\left(\frac{x^3+3}{x^5+2x^2}\right)$ using quotient rule.

Session 2 Solutions

Sol. 1 →

(a) $\lim_{x \to 5} x^2+2x+3 = 5^2+2(5)+3 = 38$

(b) $\lim_{x \to 0} \frac{x^2+x}{x} = \lim_{x \to 0} \frac{\cancel{x}(x+1)}{\cancel{x}} = 0+1 = 1$

(c) $\lim_{x \to 2} \frac{7x-14}{x^2-4x+4} = \lim_{x \to 2} \frac{7\cancel{(x-2)}}{(x-2)^{\cancel{2}}} = \lim_{x \to 2} \frac{7}{x-2} = \infty$

(d) $\lim_{x \to -3} \frac{6x+18}{(x+4)(x+3)} = \lim_{x \to -3} \frac{6\cancel{(x+3)}}{(x+4)\cancel{(x+3)}} = \frac{6}{-3+4} = 6$

Sol. 2 →

$\lim_{x \to 2} f(x) = \lim_{x \to 2} x-2 = 2-2 = 0$

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x = 0$

$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x-2 = -2$

Since $\lim_{x \to 0^-} f(x) \neq \lim_{x \to 0^+} f(x)$

$\therefore \lim_{x \to 0} f(x)$ does not exist

Sol. 3 →

(a) $\frac{df}{dx} = \frac{d}{dx}(x^2+2x+4) = \frac{d}{dx}(x^2) + \frac{d}{dx}(2x) + \frac{d}{dx}(4)$

$= 2x + 2\frac{d(x)}{dx} + 0$

$= 2x + 2$

(b) $\frac{df}{dx} = \frac{d}{dx}(x^3) = 3x^2$

Sol. 4 →

$$\frac{\partial f}{\partial x} = 3x^2 + 4x + 6xy + 3y$$

$$\frac{\partial f}{\partial y} = 3x^2 + 3x$$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} 3x^2 + 4x + 6xy + 3y \\ 3x^2 + 3x \end{bmatrix}$$

Sol. 5 →

$$\frac{\partial f}{\partial x} = 1 + z^2$$

$$\frac{\partial f}{\partial y} = 1$$

$$\frac{\partial f}{\partial z} = 2zx$$

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{bmatrix} = \begin{bmatrix} 1 + z^2 \\ 1 \\ 2zx \end{bmatrix}$$

Sol. 6 →

$$\frac{d\sigma}{dx} = \lim_{h \to 0} \frac{\sigma(x+h) - \sigma(x)}{h} = \lim_{h \to 0} \left(\frac{e^{x+h}}{1 + e^{x+h}} - \frac{e^x}{1 + e^x} \right) \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{e^{x+h}(1 + e^x) - e^x(1 + e^{x+h})}{(1 + e^{x+h})(1 + e^x)\, h}$$

$$= \lim_{h \to 0} \frac{e^{x+h} + \cancel{e^{2x+h}} - e^x - \cancel{e^{2x+h}}}{(1 + e^{x+h})(1 + e^x)\, h}$$

$$= \lim_{h \to 0} \frac{e^{x+h} - e^{x}}{(1+e^{x+h})(1+e^{x})h}$$

$$= \lim_{h \to 0} \frac{e^{x}(e^{h} - 1)}{(1+e^{x+h})(1+e^{x})h}$$

$$= \frac{e^{x}}{(1+e^{x})^2} \cdot 1 \quad \left(\because \lim_{h \to 0} \frac{e^{h}-1}{h} = 1 \text{ is given} \right)$$

$$= \frac{e^{x}}{(1+e^{x})^2}$$

Sol. 7 → $\frac{d}{dx}\left(\frac{x^3+3}{x^5+2x^2}\right) = \frac{(x^5+2x^2)(3x^2) - (x^3+3)(5x^4+4x)}{(x^5+2x^2)^2}$

$$= \frac{-2x^7 - 13x^4 - 12x}{(x^5+2x^2)^2}$$